Problem of the Week
Problem C and Solution
That's Odd

Problem

Did you know that the sum of the first n positive odd integers is n^2? The sum of the first five positive odd integers would be 5^2 or 25. We can easily check to see that $1 + 3 + 5 + 7 + 9 = 25$. When adding the first a positive odd integers to the first b positive odd integers, the sum is 180. If p is the largest odd number in the first set of numbers and q is the largest odd number in the second set of numbers, then determine the sum $p + q$.

Solution

Since there are a positive odd integers and the largest is p, then $1 + 3 + 5 + \cdots + p = a^2$.
Since there are b positive odd integers and the largest is q, then $1 + 3 + 5 + \cdots + q = b^2$.
We also know that when these two sets of odd numbers are added together, the sum is 180 so

$$(1 + 3 + 5 + \cdots + p) + (1 + 3 + 5 + \cdots + q) = a^2 + b^2 = 180.$$

One way to proceed is to pick values for a, determine a^2 and then determine if the remaining number required to sum to 180 is a perfect square. The results are summarized in the table below.

a	a^2	$b^2 = 180 - a^2$	b $(b > 0)$	Solution?
1	1	180-1=179	13.4	no
2	4	180-4=176	13.3	no
3	9	180-9=171	13.1	no
4	16	180-16=164	12.8	no
5	25	180-25=155	12.4	no
6	36	180-36=144	12	yes
7	49	180-49=131	11.4	no
8	64	180-64=116	10.8	no
9	81	180-81=99	9.9	no
10	100	180-100=80	8.9	no
11	121	180-121=59	7.7	no
12	144	180-144=36	6	yes
13	169	180-169=11	3.3	no

If $a = 14$, then $a^2 = 196$. This produces a value greater than 180 and cannot be a possible solution.

There appear to be two possible solutions. When $a = 6$ and $b = 12$, then $a^2 + b^2 = 36 + 144 = 180$. This means that adding the first 6 odd positive integers to the first 12 odd positive integers results in a sum of 180. So p is the sixth odd positive integer, namely 11, and q is the twelfth odd positive integer, namely 23. The sum, $p + q$, is $11 + 23$ or 34. The second solution, $a = 12$ and $b = 6$, produces $p = 23$ and $q = 11$. The sum, $p + q$, is still 34.